Polynomial functions as sums of periodic functions

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We prove that every real polynomial of degree n can be expressed as the sum of $n+1$ periodic functions. The proof relies on the existence of a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and therefore uses the axiom of choice. We also show that $n+1$ is optimal, and that certain non-polynomial functions (such as e^x) cannot be written as finite sums of periodic functions.

A nonconstant polynomial cannot be written as a finite sum of *continuous* periodic functions, since every continuous periodic function is bounded and a finite sum of bounded functions is bounded. In fact, an unbounded continuous function cannot be expressed as a finite sum of Lebesgue-measurable periodic functions. Consequently, nonmeasurable functions are required.

Since $\mathbb{R}$ is a vector space over $\mathbb{Q}$, the axiom of choice implies the existence of a Hamel basis $A \subset \mathbb{R}$. Thus every $x \in \mathbb{R}$ has a unique representation

$$x = \sum_{a \in A} (x, a)a,$$

where $(x, a) \in \mathbb{Q}$ for each $a \in A$, and only finitely many coefficients (x, a) are nonzero.

The coefficients (x, a) satisfy additivity in the first variable:

$$(x + y, a) = (x, a) + (y, a)$$

for all $x, y \in \mathbb{R}$ and $a \in A$, and

$$(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

Lemma 1. *If $a, b \in A$ with $a \neq b$, then the function $x \mapsto (x, a)$ is periodic with period b.*

Proof. Since $(x + b, a) = (x, a) + (b, a)$ and $(b, a) = 0$ for $a \neq b$, we obtain

$$(x + b, a) = (x, a).$$

Thus b is a period of the function $x \mapsto (x, a)$. □

Proposition 1. *The identity function $x \mapsto x$ is the sum of two periodic functions.*

Proof. Choose distinct elements $c, d \in A$. Then

$$x = (x, c)c + \sum_{a \in A \setminus \{c\}} (x, a)a.$$

Define

$$f_1(x) = (x, c)c, \qquad f_2(x) = \sum_{a \in A \setminus \{c\}} (x, a)a.$$

The function f_1 has period d, while f_2 has period c. Since

$$x = f_1(x) + f_2(x),$$

the identity function is the sum of two periodic functions. □

Proposition 2. *The function $x \mapsto x^2$ is the sum of three periodic functions.*

Proof. Using the Hamel expansion,

$$x^2 = \left(\sum_{a \in A} (x, a)a \right)^2 = \sum_{a \in A} \sum_{b \in A} (x, a)(x, b)ab.$$

Choose distinct elements $c, d, e \in A$.

The term $(x, a)(x, b)ab$ is periodic with period c unless $a = c$ or $b = c$. Similarly, terms not involving d have period d, and terms not involving e have period e.

No term involves all three of c, d, e. Therefore every term has at least one of c, d, e as a period.

Define the functions $C(x), D(x), E(x)$ as follows.

$$C(x) = \sum_{a,b \in A \setminus \{c\}} (x, a)(x, b)ab,$$

$$D(x) = (x, c)^2 c^2 + 2 \sum_{a \in A \setminus \{c,d\}} (x, a)(x, c)ac,$$

$$E(x) = 2(x, c)(x, d)cd.$$

In other words, $C(x)$ is the sum of all terms that do not involve c, $D(x)$ is the sum of all terms that involve c but do not involve d, and $E(x)$ is the sum of all terms that involve both c and d (and hence do not involve e).

Therefore

$$x^2 = C(x) + D(x) + E(x),$$

and C, D, E are periodic with periods c, d, e respectively. □

Theorem 1. *If P is a polynomial of degree n, then P is the sum of $n+1$ periodic functions.*

Proof. Using

$$x = \sum_{a \in A} (x, a)a,$$

the polynomial $P(x)$ expands into a linear combination of terms of the form

$$(x, a_1)(x, a_2) \cdots (x, a_k)$$

with $k \leq n$. Although there are infinitely many such terms, only finitely many are nonzero for each x.

Choose distinct elements

$$a_0, a_1, \cdots, a_n \in A.$$

No term in the expansion can involve all of the elements $a_0, a_1, \ldots, a_n$, since each term involves at most n distinct basis elements.

For each term T in the expansion of $P(x)$, let $i(T)$ be the least index $i \in \{0, \ldots, n\}$ such that T does not involve a_i. Such an index exists because no term involves all $n+1$ of the chosen basis elements.

For $i = 0, \ldots, n$, let $P_i(x)$ be the sum of all terms T with $i(T) = i$. Then the sums are disjoint and

$$P(x) = \sum_{i=0}^{n} P_i(x).$$

Each term in P_i avoids a_i, and therefore depends only on coefficients (x, a) with $a \neq a_i$. Since each function $x \mapsto (x, a)$ with $a \neq a_i$ is periodic with period a_i, it follows that P_i is periodic with period a_i.

Thus P is the sum of $n+1$ periodic functions. □

Theorem 2. *A polynomial of degree n cannot be written as the sum of n periodic functions.*

Proof. Suppose P is a polynomial of degree n that is the sum of n periodic functions, and assume n is minimal:

$$P(x) = f_1(x) + \cdots + f_n(x).$$

Let p be a period of f_n. Then

$$P(x+p) - P(x) = \sum_{i=1}^{n-1} \left(f_i(x+p) - f_i(x) \right).$$

The function $P(x+p) - P(x)$ is a polynomial of degree $n-1$, while each

$$g_i(x) := f_i(x+p) - f_i(x)$$

is periodic. This contradicts the minimality of n. □

Theorem 3. *The function e^x cannot be expressed as a finite sum of periodic functions.*

Proof. Suppose

$$e^x = f_1(x) + \cdots + f_n(x)$$

with n minimal and each f_i periodic.

Let p be a period of f_n. Then

$$e^x = \frac{e^{x+p} - e^x}{e^p - 1} = \sum_{i=1}^{n-1} \frac{f_i(x+p) - f_i(x)}{e^p - 1}.$$

Each term

$$g_i(x) := \frac{f_i(x+p) - f_i(x)}{e^p - 1}$$

is periodic, giving a representation with $n-1$ periodic functions, a contradiction. □