Area Bounds for a Midpoint-Crosscut Quadrilateral

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Abstract

Let $ABCD$ be a convex quadrilateral, let E, F, G, H be the midpoints of AB, BC, CD, DA, respectively, and let

$$P = AF \cap BG, \qquad Q = BG \cap CH, \qquad R = CH \cap DE, \qquad S = DE \cap AF.$$

Then $PQRS$ is a convex quadrilateral contained in $ABCD$. With the assistance of a computer algebra system, we give a short coordinate proof that

$$\frac{1}{6} < \frac{\text{area}(PQRS)}{\text{area}(ABCD)} \leq \frac{1}{5}.$$

Equality in the upper bound holds if and only if $PQRS$ is a trapezoid.

1 Introduction

A classical exercise shows that if $ABCD$ is a square, then the quadrilateral formed by the four lines AF, BG, CH, and DE has area one fifth of the area of $ABCD$. By affine invariance, the same ratio holds for every parallelogram. For a general convex quadrilateral, however, the ratio is no longer constant.

The following theorem gives the sharp bounds.

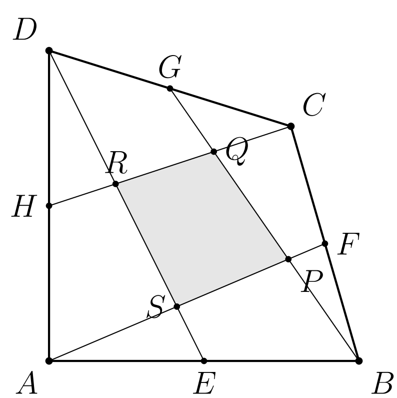

Figure 1: The midpoint-crosscut quadrilateral $PQRS$ inside $ABCD$.

Theorem 1. *Let $ABCD$ be a convex quadrilateral, let E, F, G, H be the midpoints of AB, BC, CD, DA, and let*

$$P = AF \cap BG, \qquad Q = BG \cap CH, \qquad R = CH \cap DE, \qquad S = DE \cap AF.$$

Then

$$\frac{1}{6} < \frac{\text{area}(PQRS)}{\text{area}(ABCD)} \leq \frac{1}{5}.$$

Moreover, equality in the upper bound holds if and only if $PQRS$ is a trapezoid, i.e. it has at least one pair of parallel sides.

This result was discovered previously by Rick Mabry [1].

2 Proof

Area ratios are preserved by nonsingular affine maps. Since affine maps also preserve convexity, collinearity, and midpoint relations, we may assume without loss of generality that

$$A = (0,0), \qquad B = (2,0), \qquad D = (0,2), \qquad C = (x,y),$$

where the convexity of $ABCD$ is equivalent to

$$x > 0, \qquad y > 0, \qquad x + y > 2.$$

In these coordinates,

$$E = (1,0), \qquad F = \left(\frac{x+2}{2}, \frac{y}{2}\right), \qquad G = \left(\frac{x}{2}, \frac{y+2}{2}\right), \qquad H = (0,1).$$

A routine solution of the corresponding pairs of linear equations gives

$$\begin{aligned}
P &= \left(\frac{(x+2)(y+2)}{x+3y+2}, \frac{y(y+2)}{x+3y+2}\right), \\
Q &= \left(\frac{x(x+2y)}{3x+4y-4}, \frac{(y+2)(x+2y-2)}{3x+4y-4}\right), \\
R &= \left(\frac{x}{2x+y-1}, \frac{2(x+y-1)}{2x+y-1}\right), \\
S &= \left(\frac{2(x+2)}{2x+y+4}, \frac{2y}{2x+y+4}\right).
\end{aligned}$$

The area of the outer quadrilateral is

$$\text{area}(ABCD) = x + y,$$

for instance by decomposing $ABCD$ into the triangles ABC and ACD.

Now apply the shoelace formula to P, Q, R, S. The resulting expression for area$(PQRS)$ is a rational function of x and y. Rather than display the full formula, it is more useful to consider the two differences

$$\Delta_1 := \text{area}(ABCD) - 5\,\text{area}(PQRS)$$

and

$$\Delta_2 := 6\,\text{area}(PQRS) - \text{area}(ABCD).$$

A straightforward symbolic simplification yields

$$\Delta_1 = \frac{(x+y)(x-2y+2)^2(2x+y-6)^2}{2\Pi},$$
$$\Delta_2 = \frac{5(x+y)(2x^2+xy+8y-8)(2xy+x+y^2-3y+2)}{\Pi},$$

where

$$\Pi = (x+3y+2)(2x+y-1)(2x+y+4)(3x+4y-4).$$

We now check that every factor in these expressions has the expected sign.

First, $x+y>0$, and each factor in Π is positive because

$$x+3y+2>0, \qquad 2x+y-1>x+y-1>1, \qquad 2x+y+4>0,$$

and

$$3x+4y-4=(x+y-2)+2x+3y>0.$$

Hence $\Pi>0$.

It follows immediately that $\Delta_1 \geq 0$, so

$$\text{area}(PQRS) \leq \frac{1}{5}\,\text{area}(ABCD).$$

Moreover,

$$\Delta_1 = 0 \quad \Longleftrightarrow \quad x-2y+2=0 \quad \text{or} \quad 2x+y-6=0.$$

It is routine to verify that $2x+y-6=0$ if and only if $PQ \parallel RS$, and that $x-2y+2=0$ if and only if $QR \parallel SP$. Thus equality holds precisely when $PQRS$ is a trapezoid. Moreover, the lines $2x+y-6=0$ and $x-2y+2=0$ intersect at $(2,2)$, which corresponds to the special case in which both pairs of opposite sides are parallel, so that $PQRS$ is a parallelogram.

For the lower bound, set

$$f(x,y)=2x^2+xy+8y-8, \qquad g(x,y)=2xy+x+y^2-3y+2.$$

Since

$$\frac{\partial f}{\partial x}=4x+y>0, \qquad \frac{\partial g}{\partial x}=2y+1>0,$$

both f and g are increasing functions of x for fixed y. Thus, for a given y, their minimum over the admissible region occurs on the boundary $x=\max\{0,2-y\}$.

If $0 < y \le 2$, then $x \ge 2 - y$, and therefore

$$f(x, y) \ge f(2 - y, y) = y(y + 2) > 0,$$

while

$$g(x, y) \ge g(2 - y, y) = 4 - y^2 > 0.$$

If $y \ge 2$, then $x > 0$, and hence

$$f(x, y) > f(0, y) = 8(y - 1) > 0,$$

while

$$g(x, y) > g(0, y) = (y - 1)(y - 2) \ge 0.$$

When $y = 2$, the last inequality is strict because $x > 0$ and $\partial g/\partial x > 0$. Thus $f(x, y) > 0$ and $g(x, y) > 0$ throughout the admissible region. Consequently $\Delta_2 > 0$, and so

$$\text{area}(PQRS) > \frac{1}{6}\,\text{area}(ABCD).$$

This completes the proof. For readers who wish to verify the symbolic factorization directly, Appendix A records a short `SymPy` script adapted from the original Math Stack Exchange post.

Remark 1. *The lower bound is sharp but not attained by a genuinely convex quadrilateral: in the normalized model one has*

$$\frac{\text{area}(PQRS)}{\text{area}(ABCD)} \to \frac{1}{6}$$

as $(x, y) \to (0, 2)$ *through the admissible region, corresponding geometrically to the degeneration* $C \to D$.

A `SymPy` verification script

The following script is adapted from the code included in the Math Stack Exchange answer [2]. It computes the intersection points P, Q, R, S, evaluates the shoelace areas of $ABCD$ and $PQRS$, and factors the two key expressions

$$\Delta_1 = \text{area}(ABCD) - 5\,\text{area}(PQRS), \qquad \Delta_2 = 6\,\text{area}(PQRS) - \text{area}(ABCD).$$

The output matches the formulas displayed in the proof.

```
import sympy as sp

x, y = sp.symbols("x y")
X, Y = sp.symbols("X Y")



def midpoint(p, q):
```

```python
    """Return the midpoint of the segment joining p and q."""
    return ((p[0] + q[0]) / 2, (p[1] + q[1]) / 2)


def line_through(p, q):
    """Return the equation L(X, Y) = 0 of the line through p and q."""
    return (X - p[0]) * (Y - q[1]) - (X - q[0]) * (Y - p[1])


def intersection(line1, line2):
    """Return the intersection point of the two given lines."""
    solution = sp.solve((line1, line2), (X, Y), dict=True)[0]
    return (solution[X], solution[Y])


def polygon_area(vertices):
    """Return the signed area of a polygon with ordered vertices."""
    n = len(vertices)
    return sp.simplify(
        sum(
            vertices[i - 1][0] * vertices[i][1]
            - vertices[i - 1][1] * vertices[i][0]
            for i in range(n)
        ) / 2
    )


def pretty(expr):
    """Factor and print expressions in a compact human-readable form."""
    expr = sp.factor(sp.simplify(expr))
    return str(expr).replace("**", "^").replace("*", "")


zero = sp.Integer(0)
two = sp.Integer(2)

A = (zero, zero)
B = (two, zero)
C = (x, y)
D = (zero, two)
vertices = [A, B, C, D]

E = midpoint(A, B)
F = midpoint(B, C)
G = midpoint(C, D)
H = midpoint(D, A)
```

```
P = intersection(line_through(A, F), line_through(B, G))
Q = intersection(line_through(B, G), line_through(C, H))
R = intersection(line_through(C, H), line_through(D, E))
S = intersection(line_through(D, E), line_through(A, F))
inner_vertices = [P, Q, R, S]

for label, point in zip("PQRS", inner_vertices):
    print(f"{label}x = {pretty(point[0])}")
    print(f"{label}y = {pretty(point[1])}")

outer_area = polygon_area(vertices)
inner_area = polygon_area(inner_vertices)

print(f"\nouter area = {pretty(outer_area)}")
print("inner area:")
print(pretty(inner_area))

print("\nouter area - 5 * inner area =")
print(pretty(outer_area - 5 * inner_area))

print("\n6 * inner area - outer area =")
print(pretty(6 * inner_area - outer_area))
```

References

[1] R. Mabry, *Crosscut Convex Quadrilaterals*, Mathematics Magazine **84** (2011), no. 1, 16–25.

[2] D. Radcliffe, *Quadrilateral formed by connecting the vertices of a convex quadrilateral to midpoints of non-adjacent sides*, Math Stack Exchange, answer posted November 4, 2016. Available online at Math Stack Exchange.