On the binary reverse-and-add trajectory of 10110_2

David G. Radcliffe

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Let T be the binary reverse-and-add map

$$T(n) = n + \overleftarrow{n},$$

where $\overleftarrow{n}$ denotes the integer obtained by reversing the binary expansion of n and then deleting any leading zeros. Starting from

$$a(0) = 10110_2 = 22,$$

we define

$$a(n+1) = T(a(n)), \qquad n \geq 0.$$

The first four terms are[1]

$$a(0) = 10110, \quad a(1) = 100011, \quad a(2) = 1010100, \quad a(3) = 1101001.$$

We prove that from $a(4)$ onward the binary expansions fall into a 4-cycle of explicit block form.

For a binary word w, we write 1^m and 0^m for a block of m consecutive 1's and 0's, respectively.

Theorem 1. *Let* $(a(n))_{n\geq 0}$ *be the binary reverse-and-add trajectory of* 10110*. Then*

$$a(0) = 10110, \quad a(1) = 100011, \quad a(2) = 1010100, \quad a(3) = 1101001,$$

and for every integer $k \geq 1$*,*

$$a(4k) = 10\,1^{k+1}01\,0^{k+1}, \tag{1}$$
$$a(4k+1) = 11\,0^{k-1}1000\,1^{k-1}01, \tag{2}$$
$$a(4k+2) = 10\,1^{k+1}01\,0^{k+2}, \tag{3}$$
$$a(4k+3) = 11\,0^{k+1}10\,1^{k}01. \tag{4}$$

In particular, this trajectory never produces a palindrome in binary.

We begin with the four transition identities that drive the iteration.

[1] We will write all terms in binary from here on.

Lemma 1. *For every integer $k \geq 1$, define the binary words*

$$\begin{aligned} A_k &= 10\,1^{k+1}01\,0^{k+1}, \\ B_k &= 11\,0^{k-1}1000\,1^{k-1}01, \\ C_k &= 10\,1^{k+1}01\,0^{k+2}, \\ D_k &= 11\,0^{k+1}10\,1^{k}01. \end{aligned}$$

Then

$$T(A_k) = B_k, \tag{5}$$
$$T(B_k) = C_k, \tag{6}$$
$$T(C_k) = D_k, \tag{7}$$
$$T(D_k) = A_{k+1}. \tag{8}$$

Proof. We verify the four identities by aligned binary addition. In each case, the reversed word is first written without leading zeros and then aligned with the original word.

1. Proof of (5)**.** We have

$$A_k = 10\,1^{k+1}01\,0^{k+1}.$$

Reversing the binary expansion gives

$$0^{k+1}10\,1^{k+1}01,$$

hence

$$\overleftarrow{A_k} = 10\,1^{k+1}01.$$

Aligning the two expansions yields

$$\begin{array}{rcccccccccccc} & & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & & \\ A_k & = & 1 & 0 & 1^{k-1} & 1 & 1 & 0 & 1 & 0^{k-1} & 0 & 0 \\ \overleftarrow{A_k} & = & & & & 1 & 0 & 1 & 1 & 1^{k-1} & 0 & 1 \\ \hline A_k + \overleftarrow{A_k} & = & 1 & 1 & 0^{k-1} & 1 & 0 & 0 & 0 & 1^{k-1} & 0 & 1. \end{array}$$

Thus

$$T(A_k) = A_k + \overleftarrow{A_k} = 11\,0^{k-1}1000\,1^{k-1}01 = B_k.$$

2. Proof of (6)**.** We have

$$B_k = 11\,0^{k-1}1000\,1^{k-1}01.$$

Reversing gives

$$\overleftarrow{B_k} = 10\,1^{k-1}0001\,0^{k-1}11.$$

Aligning the two expansions yields

$$\begin{array}{rccccccccccccc} & & {\scriptstyle 1} & & & & & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\ B_k & = & & 1 & 1 & 0^{k-1} & 1 & 0 & 0 & 0 & 1^{k-1} & 0 & 1 \\ \overleftarrow{B_k} & = & & 1 & 0 & 1^{k-1} & 0 & 0 & 0 & 1 & 0^{k-1} & 1 & 1 \\ \hline B_k + \overleftarrow{B_k} & = & 1 & 0 & 1 & 1^{k-1} & 1 & 0 & 1 & 0 & 0^{k-1} & 0 & 0. \end{array}$$

Thus

$$T(B_k) = B_k + \overleftarrow{B_k} = 10\,1^{k+1}01\,0^{k+2} = C_k.$$

3. Proof of (7)**.** We have

$$C_k = 10\,1^{k+1}01\,0^{k+2}.$$

Reversing gives

$$0^{k+2}10\,1^{k+1}01,$$

hence

$$\overleftarrow{C_k} = 10\,1^{k+1}01.$$

Aligning the two expansions yields

$$\begin{array}{rcccccccccc}
 & & & {\color{red}\scriptstyle 1} & {\color{red}\scriptstyle 1} & & {\color{red}\scriptstyle 1} & & & & \\
C_k & = & 1 & 0 & 1^k & 1 & 0 & 1 & 0^k & 0 & 0 \\
\overleftarrow{C_k} & = & & & & 1 & 0 & 1 & 1^k & 0 & 1 \\
\hline
C_k + \overleftarrow{C_k} & = & 1 & 1 & 0^k & 0 & 1 & 0 & 1^k & 0 & 1.
\end{array}$$

Thus

$$T(C_k) = C_k + \overleftarrow{C_k} = 11\,0^{k+1}10\,1^k01 = D_k.$$

4. Proof of (8)**.** Finally,

$$D_k = 11\,0^{k+1}10\,1^k01,$$

so

$$\overleftarrow{D_k} = 10\,1^k01\,0^{k+1}11.$$

Aligning the two expansions yields

$$\begin{array}{rccccccccccc}
 & & {\color{red}\scriptstyle 1} & & & & {\color{red}\scriptstyle 1} & & {\color{red}\scriptstyle 1} & {\color{red}\scriptstyle 1} & {\color{red}\scriptstyle 1} & \\
D_k & = & & 1 & 1 & 0^k & 0 & 1 & 0 & 1^k & 0 & 1 \\
\overleftarrow{D_k} & = & & 1 & 0 & 1^k & 0 & 1 & 0 & 0^k & 1 & 1 \\
\hline
D_k + \overleftarrow{D_k} & = & 1 & 0 & 1 & 1^k & 1 & 0 & 1 & 0^k & 0 & 0
\end{array}$$

Thus

$$T(D_k) = D_k + \overleftarrow{D_k} = 10\,1^{k+2}01\,0^{k+2} = A_{k+1}.$$

This proves all four transition identities. □

Proof of the theorem. The values

$$a(0) = 10110, \quad a(1) = 100011, \quad a(2) = 1010100, \quad a(3) = 1101001,$$

are obtained by direct computation.

We now prove (1)–(4) by induction on k.

For $k = 1$, we have

$$a(4) = 10110100 = A_1.$$

By Lemma 1,

$$\begin{aligned} a(5) &= T(a(4)) = T(A_1) = B_1, \\ a(6) &= T(a(5)) = T(B_1) = C_1, \\ a(7) &= T(a(6)) = T(C_1) = D_1. \end{aligned}$$

So the formulas hold for $k = 1$.

Now suppose that for some $k \geq 1$,

$$a(4k) = A_k, \quad a(4k+1) = B_k, \quad a(4k+2) = C_k, \quad a(4k+3) = D_k.$$

Applying the lemma gives

$$\begin{aligned} a(4k+4) &= T(a(4k+3)) = T(D_k) &&= A_{k+1}, \\ a(4k+5) &= T(a(4k+4)) = T(A_{k+1}) &&= B_{k+1}, \\ a(4k+6) &= T(a(4k+5)) = T(B_{k+1}) &&= C_{k+1}, \\ a(4k+7) &= T(a(4k+6)) = T(C_{k+1}) &&= D_{k+1}. \end{aligned}$$

Thus the formulas hold with k replaced by $k+1$.

By induction, (1)–(4) hold for all $k \geq 1$. This completes the proof. $\square$

The proof shows that after $a(3)$ the iteration is governed by the cycle

$$A_k \longmapsto B_k \longmapsto C_k \longmapsto D_k \longmapsto A_{k+1},$$

so that each full cycle increases the parameter k by 1.